Exercise 5E

1 a i $r = 0.1$ so the series is convergent as $|r| < 1$

ii $S_\infty = \frac{1}{1-0.1} = \frac{10}{9}$

b $r = 2$ so the series is not convergent as $|r| \geqslant 1$.

c i $r = -0.5$ so the series is convergent as $|r| < 1$.

ii $S_\infty = \frac{10}{1+0.5} = \frac{20}{3} = 6\frac{2}{3}$

d This is an arithmetic series and so does not converge.

e $r = 1$ so the series is not convergent as $|r| \geqslant 1$.

f i $r = \frac{1}{3}$ so the series is convergent as $|r| < 1$

ii $S_\infty = \frac{3}{1-\frac{1}{3}} = \frac{9}{2} = 4\frac{1}{2}$

g This is an arithmetic series and so does not converge.

h i $r = 0.9$ so the series is convergent as $|r| < 1$.

ii $S_\infty = \frac{9}{1-0.9} = 90$

2 $a = 10$, $S_\infty = 30$

$$\frac{10}{1-r} = 30$$
$$10 = 30(1-r)$$
$$30r = 20$$
$$r = \frac{2}{3}$$

3 $a = -5$, $S_\infty = -3$

$$\frac{-5}{1-r} = -3$$
$$-5 = -3(1-r)$$
$$3r = -2$$
$$r = -\frac{2}{3}$$

4 $S_\infty = 60$, $r = \frac{2}{3}$

$$\frac{a}{1-\frac{2}{3}} = 60$$
$$\frac{a}{\frac{1}{3}} = 60$$
$$a = 20$$

5 $S_\infty = 10$, $r = -\frac{1}{3}$

$$\frac{a}{1+\frac{1}{3}} = 10$$
$$\frac{a}{\frac{4}{3}} = 10$$
$$a = \frac{40}{3} = 13\frac{1}{3}$$

6 $0.\dot{2}\dot{3}\ldots = \frac{23}{100} \xrightarrow{\times\frac{1}{100}} \frac{23}{10\,000} \xrightarrow{\times\frac{1}{100}} \frac{23}{1\,000\,000} + \ldots$

This is an infinite geometric series:

$a = \frac{23}{100}$ and $r = \frac{1}{100}$.

Use $S_\infty = \frac{a}{1-r}$.

$$0.\dot{2}\dot{3}\ldots = \frac{\frac{23}{100}}{1-\frac{1}{100}} = \frac{\frac{23}{100}}{\frac{99}{100}}$$
$$= \frac{23}{100} \times \frac{100}{99} = \frac{23}{99}$$

7 $S_3 = 9,\ S_\infty = 8$

$S_3 = \dfrac{a(1-r^3)}{1-r} = 9$ **(1)**

$S_\infty = \dfrac{a}{1-r} = 8$ **(2)**

$8(1-r^3) = 9$ (substituting **(2)** into **(1)**)

$1 - r^3 = \dfrac{9}{8}$

$r^3 = -\dfrac{1}{8}$

$r = -\dfrac{1}{2}$

$a = 8\left(1 + \dfrac{1}{2}\right)$ (from **(2)**)

$a = 12$

8 a $a = 1, r = -2x$

As the series is convergent, $|-2x| < 1$

If $x < 0$ then $2x < 1$, so $x < \dfrac{1}{2}$;

if $x > 0$ then $-2x < 1$, so $x > -\dfrac{1}{2}$

Hence, $-\dfrac{1}{2} < x < \dfrac{1}{2}$.

b $S_\infty = \dfrac{1}{1+2x}$

9 a $a = 2,\ S_\infty = 16 \times S_3$

$S_3 = \dfrac{2(1-r^3)}{1-r}$

$16 \times \dfrac{2(1-r^3)}{1-r} = \dfrac{2}{1-r}$

$32(1-r^3) = 2$

$r^3 = \dfrac{15}{16}$

$r = 0.9787$

b $u_4 = ar^3 = 2 \times 0.9787^3 = 1.875$

10 a $a = 30,\ S_\infty = 240$

$\dfrac{30}{1-r} = 240$

$\dfrac{1}{8} = 1 - r$

$r = \dfrac{7}{8}$

b $u_4 - u_5 = ar^3 - ar^4$

$= 30\left(\dfrac{7}{8}\right)^3 - 30\left(\dfrac{7}{8}\right)^4$

$= 2.51$

c $S_4 = \dfrac{30\left(1 - \left(\dfrac{7}{8}\right)^4\right)}{1 - \dfrac{7}{8}}$

$= 99.3$

d If $S_n = \dfrac{30\left(1 - \left(\dfrac{7}{8}\right)^n\right)}{1 - \dfrac{7}{8}} = 180$

$\dfrac{30\left(1 - \left(\dfrac{7}{8}\right)^n\right)}{\dfrac{1}{8}} = 180$

$1 - \left(\dfrac{7}{8}\right)^n = 0.75$

$0.875^n = 0.25$

$n = \dfrac{\log 0.25}{\log 0.875}$

$n = 10.38$

$n = 11$

11 a $ar = \dfrac{15}{8},\ S_\infty = 8$

$\dfrac{a}{1-r} = 8$

$a = 8(1-r)$

$a = \dfrac{15}{8r}$

$\dfrac{15}{8r} = 8(1-r)$

$15 = 64r - 64r^2$

$64r^2 - 64r + 15 = 0$

11 b $(8r-3)(8r-5)=0$

$r=\frac{3}{8}$ or $r=\frac{5}{8}$

c When $r=\frac{3}{8}$

$a=8\left(1-\frac{3}{8}\right)=5$

When $r=\frac{5}{8}$

$a=8\left(1-\frac{5}{8}\right)=3$

d $r=\frac{3}{8}$

If $S_n=\dfrac{5\left(1-\left(\frac{3}{8}\right)^n\right)}{1-\frac{3}{8}}=7.99$

$\dfrac{5\left(1-\left(\frac{3}{8}\right)^n\right)}{\frac{5}{8}}=7.99$

$1-0.375^n=0.99875$

$0.375^n=0.00125$

$n=\dfrac{\log 0.00125}{\log 0.375}$

$n=6.815$

$n=7$

Challenge

a First series $a+ar+ar^2+ar^3+\ \ldots$

Second series $a^2+a^2r^2+a^2r^4+a^2r^6+\ \ldots$

The second series is geometric with common ratio is r^2 and first term a^2.

b $\dfrac{a}{1-r}=7\Rightarrow a=7(1-r)\Rightarrow a^2=49(1-r)^2$

$\dfrac{a^2}{1-r^2}=35\Rightarrow\dfrac{49(1-r)^2}{(1+r)(1-r)}=35$

$49(1-r)=35(1+r)$

$49-49r=35+35r$

$84r=14$

$r=\frac{1}{6}$